ON THE ARITHMETIC SELF-INTERSECTION NUMBER OF THE DUALIZING SHEAF ON ARITHMETIC SURFACES

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Abstract. We study the arithmetic self-intersection number of the dualizing sheaf on arithmetic surfaces with respect to morphisms of a particular kind. We obtain upper bounds for the arithmetic self-intersection number of the dualizing sheaf on minimal regular models of the modular curves associated with congruence subgroups $\Gamma_0(N)$ with square free level, as well as for the modular curves $X(N)$ and the Fermat curves with prime exponent.

Contents

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Introduction

0.1. Let X be a curve defined over some number field K such that its genus g is larger than one. In the arithmetic intersection theory on a regular model $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of X the hermitian line bundle $\overline{\omega}_{\mathrm{Ar}}$, where $\overline{\omega}_{\mathrm{Ar}}$ is the dualizing sheaf $\omega_{\mathcal{X}} = \omega_{\mathcal{X}/\mathcal{O}_K} \otimes f^*\omega_{\mathcal{O}_K/\mathbb{Z}}$ equipped with the Arakelov metric (see [Ar], p.1177, [MB1], p.75), plays a prominent role. Aspects of the arithmetic meaning of its arithmetic self-intersection number $\overline{\omega}^2_{\mathrm{Ar}}$ are addressed in [Sz], [Ul] and [Zh]. In particular, certain conjectured upper bounds for algebraic families of arithmetic surfaces (see e.g. [La], p.166, or [Sz], p.244) would imply a effective version of Mordell's conjecture (cf. [Pa], [MB2], or Vojta's appendix in [La]). Thus such bounds are equivalent to the uniform *abc*-conjecture for number fields [Fr], which in turn has lots of different applications and consequences in number theory.

Except for the discrete series of modular curves $X_0(N)$, there are only few results on estimates or even upper bounds for $\overline{\omega}^2_{\mathrm{Ar}}$. More precisely, for the modular curves $X_0(N)$ with N square free and $(N, 6) = 1$ one has the following asymptotic formula

$$\overline{\omega}^2_{\mathcal{X}_0(N),\mathrm{Ar}} = 3g_N \log(N)(1 + O(\log\log(N)/\log(N))). \tag{0.1.1}$$

In [AU] Abbes and Ullmo proved a lower bound and they derived with methods, which depend strongly on the specific arithmetic of $\Gamma_0(N)$, two different formulae for $\overline{\omega}^2_{\mathcal{X}_0(N),\mathrm{Ar}}$, which contained certain quantities that they couldn't estimate. In [MU] Michel and Ullmo provided estimates for certain integrals of Eisenstein series against the canonical volume form. These estimates together with a calculation of the Neron-Tate height of certain Heegner divisors proved the formula (0.1.1) by using one of the formulae of [AU]. In [JK1] Jorgenson and Kramer obtained estimates for derivatives of the Selberg zeta functions, which by means of the other formula of [AU] also yields this estimate.

0.2. The main result of this article is a new formula for $\overline{\omega}^2_{\mathrm{Ar}}$, see theorem 4.2 below. As consequences of this rather technical result upper bounds for $\overline{\omega}^2_{\mathrm{Ar}}$ for particular curves may be calculated in a straightforward manner. Prominent examples are modular curves $X(\Gamma)$ associated with (congruence) subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$ and Fermat curves. Bounds for these curves have been asked for since the beginning of Arakelov theory (see e.g. [La], p. 130 or [MB2], 8.2).

Compared to the works [AU] plus [MU] or [JK1] this new method yields a new and considerably shorter proof of an upper bound for the curves $X_0(N)$. In addition it is not restricted to square-free levels N. The computations for the geometric contribution to $\overline{\omega}^2_{\mathrm{Ar}}$ are standard and similar to those of Abbes-Ullmo [AU]. However, for the analytic contributions, we are able to employ recent results on the sup norm of automorphic forms due to Jorgenson-Kramer [JK2], thus by-passing most of the technical difficulties in the original Abbes-Ullmo approach. A key new ingredient is a systematic use of the generalised arithmetic intersection theory [Kü].

0.3. We now set the assumptions for the next theorem which is central for our applications. Let $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface and write Y for its generic fiber. Let $\infty, P_1, ..., P_r \in Y(K)$ such that $Y \setminus \{\infty, P_1, ..., P_r\}$ is hyperbolic. Then we consider any arithmetic surface $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ equipped with a morphism of arithmetic surfaces $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ such that the induced morphism $\boldsymbol{\beta} : X \to Y$ of algebraic curves defined over K is unramified above $Y(K) \setminus \{\infty, P_1, ..., P_r\}$. Let $g \geq 2$ be the genus of X and $d = \deg(\boldsymbol{\beta})$. We write $\boldsymbol{\beta}^*\infty = \sum b_j S_j$ and the points S_j will be called cusps. Set $b_{\max} = \max_j\{b_j\}$. Divisors on X with support in the cusps of degree zero are called cuspidal. Finally, a prime $\mathfrak{p}$ is said to be bad if the fiber of $\mathcal{X}$ above $\mathfrak{p}$ is reducible[1].

A consequence of our main results is the following

Theorem I. *Let $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ be a morphism of arithmetic surfaces as in 0.3. Assume that all cusps are K-rational points and that all cuspidal divisors are torsion, then the arithmetic self-intersection number of the dualizing sheaf on $\mathcal{X}$ satisfies the inequality*

$$(0.3.1) \quad \overline{\omega}_{Ar}^2 \leq (2g-2)\left(\log|\Delta_{K|\mathbb{Q}}|^2 + [K:\mathbb{Q}]\,(\kappa_1 \log b_{\max} + \kappa_2) + \sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \operatorname{Nm}(\mathfrak{p})\right),$$

where $\kappa_1 = \kappa_1(Y, \infty, P_1, ..., P_r) \in \mathbb{R}$ is a constant independent of $\mathcal{X}$; similarly the constant in κ_2 is independent of $\mathcal{X}$. The coefficients $a_{\mathfrak{p}} \in \mathbb{Q}$ are determined by certain local intersection numbers (see formula (0.3.2) *below).*

To keep the notation simple, we write S_j also for the Zariski closure in $\mathcal{X}$ of a cusp S_j. Let $\mathcal{K}$ be a canonical divisor of $\mathcal{X}$, then for each cusp S_j we can find a divisor $\mathcal{F}_j$ such that

$$\left(S_j + \mathcal{F}_j - \frac{1}{2g-2}\mathcal{K}\right).\,\mathcal{C}_l^{(\mathfrak{p})} = 0$$

for all irreducible components $\mathcal{C}_l^{(\mathfrak{p})}$ of the fiber $f^{-1}(\mathfrak{p})$ above $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$. Similarily we find for each cusp S_j a divisor $\mathcal{G}_j$ such that also for all $\mathcal{C}_l^{(\mathfrak{p})}$ as before

$$\left(S_j + \mathcal{G}_j - \frac{1}{d}\boldsymbol{\beta}^*\infty\right).\,\mathcal{C}_l^{(\mathfrak{p})} = 0.$$

Then the rational numbers $a_{\mathfrak{p}}$ in the theorem are determined by the following arithmetic intersection numbers of trivially metrised hermitian line bundles

$$(0.3.2) \qquad \sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \operatorname{Nm}(\mathfrak{p}) = -\frac{2g}{d}\sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 + \frac{2g-2}{d}\sum_j b_j\, \mathcal{O}(\mathcal{F}_j)^2.$$

If $\mathcal{X}$ is a suitable model of a modular curve associated with a congruence subgroup Γ, then because of the Manin-Drinfeld theorem (see e.g. [El]) the assumptions of theorem I are satisfied for the natural morphism given by the j-map. In particular if Γ is of certain kind,

[1]note that a prime of bad reduction need not be a bad prime

then the coefficients $a_{\mathfrak{p}}$ in (0.3.1) can be computed explicitly by means of the descriptions of models for $X(\Gamma)$ (see e.g. [KM], [DR]). We illustrate this with the following theorem.

Theorem II. *Let N be a square free integer having at least two different prime factors and $(N,6)=1$. Let $\mathcal{X}_0(N)$ be the minimal regular model of the modular curve $X_0(N)$ and g_N its genus. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \le \kappa\, g_N \log(N) + O(g_N)$$

where $\kappa \in \mathbb{R}$ is a positive constant independent of N.

We point to the fact that our upper bound together with the lower bound in [AU] implies the asymptotic $\overline{\omega}^2_{\mathrm{Ar}} \sim g_N \log(N)$. In contrast to the previously known results the above bound is easily deduced from the results in this paper.

In forthcoming papers [C],[CK] we explicitly calculate the coefficients $a_{\mathfrak{p}}$ for Fermat curves and other modular curves.

0.4. Since the calculations of the quantities $a_{\mathfrak{p}}$ in Theorem I are rather involved, we consider instead for each bad prime $\mathfrak{p}$ the quantity $b_{\mathfrak{p}}$, which is defined as follows. Let

$$\mathcal{X} \times \overline{\mathbb{F}}_{\mathfrak{p}} = \sum_{j=1}^{r_{\mathfrak{p}}} m_j C_j^{(\mathfrak{p})}$$

be the decomposition in irreducible components and set

$$u_{\mathfrak{p}} = \max_{i,j} |C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}|, \qquad l_{\mathfrak{p}} = \min_{C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})} \neq 0} |C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}|.$$

We further denote by $c_{\mathfrak{p}}$ the connectivity of the dual graph of $\mathcal{X} \times \overline{\mathbb{F}}_{\mathfrak{p}}$, i.e. minimal number of intersection points needed to connect any two irreducible components. With these informations we can define

$$b_{\mathfrak{p}} = \left(\sum_{k=1}^{c_{\mathfrak{p}}} \left(\sum_{l=1}^{k} \left(\frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}}\right)^{l-1} \right)^2 + (r_{\mathfrak{p}} - c_{\mathfrak{p}} - 1) \left(\sum_{l=1}^{c_{\mathfrak{p}}} \left(\frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}}\right)^{l-1} \right)^2 \right) \frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}^2}$$

Theorem III. *The rational numbers $a_{\mathfrak{p}}$ in Theorem I satisfy $a_{\mathfrak{p}} \le 2g\, b_{\mathfrak{p}}$.*

For the next results the quantities $b_{\mathfrak{p}}$ are used,

Theorem IV. *Let N be an integer having at least two different prime factors and $(N,6)=1$. Let $\mathcal{X}(N)$ be the regular model of the modular curve $X(N)$ given by the moduli description (see* [KM]*). Then the arithmetic self-intersection number of its dualizing sheaf equipped*

with the Arakelov metric is bounded from above by

$$\overline{\omega}^2_{Ar} \leq (2g_N - 2)\left(\log|\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_N):\mathbb{Q}](\kappa_1 \log N + \kappa_2) + 2\sum_{\mathfrak{p}\supset(N)} b_{\mathfrak{p}} \log \mathrm{Nm}\,\mathfrak{p}\right),$$

where g_N denotes the genus of $X(N)$, $\kappa_1, \kappa_2 \in \mathbb{R}$ are constants independent of N and

$$b_{\mathfrak{p}} \leq \frac{(r_{\mathfrak{p}}-1)^2 m_{\mathfrak{p}}}{s_{\mathfrak{p}}},$$

where $r_{\mathfrak{p}}$ is as above, $m_{\mathfrak{p}} \leq p^{2v_p(N)-2}$ where $(p) = \mathfrak{p} \cap \mathbb{Z}$ and $s_{\mathfrak{p}}$ is number of supersingular points above $\mathfrak{p}$.

Other examples of curves where our result could be applied are the Fermat curves. Here we consider just the Fermat curves with prime exponents.

Theorem V. *Let $\mathcal{X}$ be the desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p+y^p=z^p$ with prime exponent p (see* [Mc]*). Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \leq (2g_p - 2)\left(\log|\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_p):\mathbb{Q}](\kappa_1 \log p + \kappa_2) + g_p\, p^7 \log p\right)$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ are constants independent of p.

We finally observe that there are only finitely many isomorphism classes of curves provided with a morphism of a fixed degree and fixed branch points. Therefore no continuous family of curves satisfying the assumptions of theorem I exists.

0.5. Plan of paper. In the first and second sections of this paper we present the necessary background material on Arakelov theory on arithmetic surfaces. After this preparatory work we study in the third section the behavior of arithmetic intersection numbers with respect to a finite morphism. The fourth section is devoted to the analytical aspects needed in our bound (0.3.1). In section five we study the geometric aspects used in the bound of the quantities $a_{\mathfrak{p}}$ of (0.3.1). In the last three sections we apply our result to the aboves mentioned examples.

1. Basic properties of Green functions.

We recollect here some basic facts related to normalised Green functions on a compact Riemann surface X. This is mainly due to the fact that inconsistent normalizations for these Green functions are used in the literature. Our normalization is as in [SABK], [La] and [Kü]; it is twice of the green function advocated in [BKK1] and it is -2 times the normalization used in [El2], [Sz] and [Fa].

1.1. Volume forms. Let ν be a volume form (c.f. [La] p.21), i.e. a smooth, positive, real $(1,1)$-form with $\int_X \nu(z) = 1$. Given two volume forms μ, ν we can write $\mu(z) = f(z)\nu(z)$, where f is a smooth function.

If $\boldsymbol{\beta} : X \to Y$ is a morphism of compact Riemann surfaces and if μ is a volume form on Y, then $(\deg \boldsymbol{\beta})^{-1}\, \boldsymbol{\beta}^*\mu$ need not to be volume form anymore. Indeed the ramification forces the pull-back form to vanish. On the other hand it may happen that the pull-back of a singular form becomes a volume form.

1.2. Green functions. A Green function g associated with ν is a real valued function on $X \times X$ which is smooth outside the diagonal and has near the diagonal an expansion $g(z,w) = -\log|z-w|^2 + h(z,w)$ with a smooth function h. As a current it satisfies

$$\mathrm{dd}^c\,[g(w,z)] + \delta_w = [\nu(z)] \tag{1.2.1}$$

and it is called normalized if in addition for all $w \in X$

$$\int_X g(z,w)\nu(z) = 0. \tag{1.2.2}$$

One can show that there is a unique normalized Green function g_ν associated to ν.

1.3. Resolvent Kernel. Normalized Green functions are also referred to as the resolvent kernel. In order to describe this we consider the space $\mathcal{C}^\infty(X,\mathbb{C})$ of complex valued $\mathcal{C}^\infty$-functions on X. Then the Laplace operator $\Delta = \Delta_\nu$ associated with a volume form ν is defined by

$$\Delta f \cdot \nu = \frac{1}{\pi i}\partial\bar\partial f = -\,\mathrm{dd}^c f, \tag{1.3.1}$$

here $f \in \mathcal{C}^\infty(X,\mathbb{C})$. The eigenvalues of Δ are positive reel numbers $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots$ and $\lim_{m\to\infty}\lambda_m = \infty$. We let $\phi_0 = 1,\ \phi_1,\ \ldots$ denoted the corresponding normalized eigenfunctions, i.e., $\int_X \phi_n\overline{\phi}_m\nu = \delta_{n,m}$. If in addition $f \in L^2(X,\nu) \cap \mathcal{C}^\infty(X,\mathbb{C})$, then f has an expansion $f(z) = \sum_{m\geq 0} a_m\phi_m(z)$ with $a_m = \int_X f(u)\overline{\phi}_m(u)\nu(u)$. With the above normalisation we have the following results.

1.4. Theorem. *For $f(z) = \sum_{m\geq 0} a_m\phi_m(z) \in L^2(X,\nu)\cap\mathcal{C}^\infty(X,\mathbb{C})$ we set*

$$\tilde G_\nu(f)(z) = \sum_{m>0}\frac{a_m}{\lambda_m}\phi_m(z).$$

Then we have

$$\tilde G_\nu(f)(z) = \int_X g_\nu(z,w)f(w)\nu(w).$$

Proof. This fact is well-known, see e.g. [El2], p.94, or [Fa], p.394. For the convenience of the reader we recall the proof in our normalisation. It suffices to show that for all

$f \in L^2(X,\nu) \cap \mathcal{C}^\infty(X,\mathbb{C})$ with $\int_X f(z)\nu(z) = 0$ we have the equality

$$f(P) = \int_X g_\nu(P,z)\Delta f(z)\nu(z),$$

since the kernel of Δ is spanned by the constant function 1. Using the identity (1.3.1) and the Green's equation (1.2.1) we derive

$$\begin{aligned}\int_X g_\nu(P,z)\Delta f(z)\nu(z) &= -\int_X g_\nu(P,z)\,\mathrm{dd}^c f(z) = -\,\mathrm{dd}^c[g_\nu(P,z)](f) \\ &= \delta_P(f) - [\nu](f) = f(P).\end{aligned}$$

$\square$

1.5. Corollary. *If in addition* $\int_X f(z)\nu(z) = 0$*, then* $\mathrm{dd}^c - \int_X g_\nu(z,w)f(w)\nu(w) = f(z)\nu(z)$*.*

Proof. By theorem 1.4 we have $\Delta\tilde{G}_\nu(f)(z) = f(z)$. Now using the identity (1.3.1) we deduce

$$\mathrm{dd}^c\left(-\int_X g_\nu(z,w)f(w)\nu(w)\right) = \Delta\tilde{G}_\nu(f)(z)\cdot\nu(z) = f(z)\nu(z).$$

$\square$

1.6. Corollary. *Let* $g_\nu(z,w)$*,* $g_\mu(z,w)$ *be the normalized Green functions associated with the volume forms* ν*,* μ *respectively. Then*

$$g_\nu(z,w) = g_\mu(z,w) + a_{\mu,\nu}(z) + a_{\mu,\nu}(w) + c_{\mu,\nu}, \tag{1.6.1}$$

where for $z \in X$ *we set*

$$a_{\mu,\nu}(z) = -\int_X g_\mu(z,u)\nu(u), \tag{1.6.2}$$

$$c_{\mu,\nu} = -\int_X a_{\mu,\nu}(w)\nu(w) = \int_{X\times X} g_\mu(w,u)\nu(u)\nu(w). \tag{1.6.3}$$

Proof. The right hand is except of a logarithmic singularity along the diagonal in $X \times X$ smooth. It is also orthogonal to ν in both variables. The positivity assumption on μ allow us to write $\nu(u) = (f(u)+1)\cdot\mu(u)$ with $f \in L^2(X,\nu) \cap \mathcal{C}^\infty(X,\mathbb{C})$. Observe that $\int_X f(u)\mu(u) = 0$ and since g_μ is normalized we have $a_{\mu,\nu}(z) = -\int_X g_\mu(z,u)f(u)\mu(u)$. From corollary 1.5 we deduce $\mathrm{dd}^c\, a_{\mu,\nu}(z) = f(z)\mu(z) = \nu(z) - \mu(z)$, hence the right hand side equals by unicity g_ν. $\square$

1.7. Lemma. *Let* ν*,* μ *be volume forms on* X*, then for any* $P \in X$

$$\int_X (g_\nu(z,P) - g_\mu(z,P))\cdot(\mu(z)+\nu(z)) = 2\,a_{\mu,\nu}(P) + c_{\mu,\nu}.$$

Proof. By means of corollary 1.6 we have $g_\nu(z,P) - g_\mu(z,P) = a_{\mu,\nu}(z) + a_{\mu,\nu}(P) + c_{\mu,\nu}$. By changing the order of integration we derive the identity

$$\int a_{\mu,\nu}(z)\,(\mu(z)+\nu(z)) = \int\left(\int -g_\mu(z,w)\nu(w)\right)(\mu(z)+\nu(z)) = -c_{\mu,\nu},$$

thus, since $\int_X (\mu(z)+\nu(z)) = 2$, we obtain the claim. □

1.8. Lemma. *Let ν, μ be volume forms on X and write $\nu(z) = (f(z)+1)\cdot\mu(z)$, then we have*

$$0 \le c_{\mu,\nu} \le \frac{2}{\lambda_1}||f||^2_{L^2}, \tag{1.8.1}$$

here λ_1 is the first non zero eigenvalue of Δ_μ and $||f||^2_{L^2} = \int_X f(z)\overline{f(z)}\mu(z)$.

Proof. We first observe $f(z)$ is a smooth, square integrable function that is orthogonal to the constants with respect to μ. By theorem 1.4 we have since $f(z) = \overline{f(z)}$ the upper bound

$$c_{\mu,\nu} = \int_X \tilde{G}_\mu(f)(z)\overline{f(z)}\mu(z) = \sum_{m>0} 2\frac{|a_m|^2}{\lambda_m} \le \frac{2}{\lambda_1}\sum_{m>0}|a_m|^2 = \frac{2}{\lambda_1}\int_X f(z)\overline{f(z)}\mu(z),$$

the lower bounds is now obvious. □

2. Hyperbolic Green function

In the sequel we need generalisations of the basic properties of Green functions associated with smooth volume forms presented in section 1. We have to consider Green functions associated with the logarithmically singular hyperbolic metric also.

2.1. Hyperbolic curves over $\mathbb{C}$. Let $\mathbb{H} := \{z = x+iy \in \mathbb{C} \,|\, \mathrm{Im}\, z = y > 0\}$ denote the upper half plane. We fix a fuchsian subgroup $\Gamma_\circ \subset \mathrm{PSL}_2(\mathbb{R})$ of the first kind. Then for any subgroup Γ of finite index in $\Gamma_\circ$ the quotient $\Gamma\setminus\mathbb{H}$ by the natural action of Γ on $\mathbb{H}$ has the structure of a Riemann surface, which can be compactified by adding finitely many cusps. We denote this compact Riemann surface by $X(\Gamma)$ and we call $X(\Gamma)$ a hyperbolic curve.

On $X(\Gamma)$ there may be also finitely many elliptic points, which correspond to $z \in \mathbb{H}$ with a non-trivial isotropy subgroup in Γ. It is common to identify a local coordinate z on $X(\Gamma)$ with a preimage $z \in \mathbb{H}$.

The inclusion of a subgroup Γ of finite index in $\Gamma_\circ$ induces a finite, holomorphic map $f : X(\Gamma) \longrightarrow X(\Gamma_\circ)$ of degree $\deg f = [\Gamma_\circ : \Gamma]$. This morphism of compact Riemann surfaces is ramified only above the elliptic points and the cusps of $X(\Gamma_\circ)$. For more details, cf. [Sh].

2.2. Hyperbolic metric on $X(\Gamma)$. The $(1,1)$-form associated to the hyperbolic metric on $\mathbb{H}$ is given by

$$\mu = \frac{dx \wedge dy}{y^2} = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{(\operatorname{Im} z)^2}. \tag{2.2.1}$$

By abuse of notation we denote the $(1,1)$-form induced by μ on $X(\Gamma)$ also by μ. It is singular and its singularities occur at the elliptic fixed points and at the cusps of $X(\Gamma)$ (see e.g. [Kü], p.222). We write μ_Γ for the log-log singular volume form on $X(\Gamma)$ determined by μ; i.e. $\int_{X(\Gamma)} \mu_\Gamma = 1$. The following facts are crucial for our purpose. If ν is a volume form on $X(\Gamma)$, then we can write $\nu = f(z)\mu_\Gamma$, where $f(z)$ is a smooth function on $X(\Gamma)$. If $f : X(\Gamma) \to X(\Gamma_\circ)$ is a morphism of hyperbolic curves, then $f^*\mu_{\Gamma_\circ} = \deg(f)\mu_\Gamma$.

2.3. Spectral theory. The space of square integrable functions with respect to μ_Γ on $X(\Gamma)$ will be denoted by $L^2(X(\Gamma), \mu_\Gamma)$. Any smooth differentiable function $f(z) \in L^2(X(\Gamma), \mu_\Gamma)$ has the following spectral decomposition

$$f(z) = \sum_{n\geq 0} a_n \phi_n(z) + \sum_\kappa \int_0^\infty h_\kappa(t) E_\kappa(z, \frac{1}{2} + it)\mathrm{d}t$$

with $\{\phi_n(z)\}$ an orthonormal basis of eigenfunctions for the discrete spectrum of the hyperbolic laplacian $\Delta_\Gamma = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ and $\{E_\kappa\}$ a complete set of Eisenstein series for the cusps κ of Γ. One has

$$\|f(z)\|^2_{L^2} = \sum_{n\geq 0} |a_n|^2 + 2\pi \sum_\kappa \int_0^\infty |h_\kappa(t)|^2 \mathrm{d}t.$$

2.4. Hyperbolic Green function. The automorphic Green function is defined for $\operatorname{Re} s > 1$ by the convergent series

$$g_\Gamma(z_1, z_2; s) := 2\pi \cdot \sum_{\gamma\in\Gamma} -2Q_{s-1}\left(1 + \frac{|z_1 - \gamma z_2|^2}{2y_1 \operatorname{Im}(\gamma z_2)}\right).$$

A well-known theorem, see e.g.[Fay] Thm. 2.3, states that we have meromorphic continuation in s and that in $s = 1$ the Laurent development is given by

$$g_\Gamma(z_1, z_2, s) = \frac{3/\pi}{[\Gamma : \Gamma(1)]} \frac{1}{s(s-1)} + g_\Gamma(z_1, z_2) + O(s-1),$$

where the $O(s-1)$ term is smooth in z_1, z_2. We call the constant Term at $s = 1$ the *hyperbolic Green function* for $X(\Gamma)$. By construction the function $g_\Gamma(z_1, z_2)$ is Γ-invariant and therefore descends to a function on $X(\Gamma)$. It is well known that $g_\Gamma(z_1, z_2)$ has a logarithmic singularity along the diagonal and that it is smooth outside the cusp and the elliptic fixed points, where it has some mild loglog singularities. Another construction of it by means of differentials of the third kind was given by W. Roelke [Roe], p. 22, who

showed that outside the cusps of $X(\Gamma)$ indeed $g_\Gamma(z_1, z_2)$ is the normalised Green function for μ_Γ.

2.5. Lemma. *Let $g_\nu(z, w)$ be the normalized Green function associated with a volume form ν and let $g_\Gamma(z, w)$ be the hyperbolic Green function. Then if P is a cusp of $X(\Gamma)$ we have for the Green function considered as a function in z the equality*

$$g_\nu(z, P) = \widetilde{g}_\Gamma(z, P; w_0) + a_{\mu_\Gamma,\nu}(z) + \widetilde{a}_{\mu_\Gamma,\nu;w_0}(P) + c_{\mu_\Gamma,\nu}$$

where we set with the choice of a point w_0 away from the cusps

$$\begin{aligned} \widetilde{g}_\Gamma(z, P; w_0) &= \lim_{w\to P} \left(g_\Gamma(z, w) - g_\Gamma(w_0, w)\right), \\ a_{\mu_\Gamma,\nu}(z) &= -\int_X g_\Gamma(z, u)\nu(u), \\ \widetilde{a}_{\mu_\Gamma,\nu,w_0}(P) &= \lim_{w\to P} \left(a_{\mu_\Gamma,\nu}(w) + g_\Gamma(w_0, w)\right), \\ c_{\mu_\Gamma,\nu} &= \int_{X\times X} g_\Gamma(v, w)\nu(w)\nu(v). \end{aligned}$$

Proof. It is well known that corollary 1.5 also holds with μ_Γ instead of a volume form ν, see e.g. [Roe], p. 35. From this it is clear that analogous to 1.6 we have with notation as above for $z, w \in X(\Gamma) \setminus \{\text{cusps}\}$

$$g_\nu(z, w) = g_\Gamma(z, w) + a_{\mu_\Gamma,\nu}(z) + a_{\mu_\Gamma,\nu}(w) + c_{\mu_\Gamma,\nu}.$$

By unicity of $g_\nu(z, w)$ the right hand side is a smooth function for all pairs z, w outside the diagonal. The loglog singularity of $g_\Gamma(z, w)$ for w approaching a cusp P is independent of z, see e.g. [Fay], p.161. Therefore the functions $\widetilde{g}_\Gamma(z, P; w_0)$ and $\widetilde{a}_{\mu_\Gamma,\nu,w_0}(P)$ are well-defined. Again by unicity of $g_\nu(z, w)$ the claim follows. □

2.6. Remark. In [AU] versions of Theorem 1.4, its Corollaries 1.5, 1.6 and Lemma 1.8 are used to express the Green function for the canonical metric evaluated at two different cusps with the Green function for the hyperbolic metric on the modular curve $X_0(N)$. In this article we have as a simplification to [AU] only to consider the evaluation at one cusp instead of two.

2.7. Lemma. *Let ν be a volume form on $X(\Gamma)$, then for a cusp $P \in X(\Gamma)$*

$$\int_X \left(g_\nu(z, P) - \widetilde{g}_\Gamma(z, P; w_0)\right) \cdot \left(\mu_\Gamma(z) + \nu(z)\right) = 2\,\widetilde{a}_{\mu_\Gamma,\nu;w_0}(P) + c_{\mu_\Gamma,\nu}.$$

Proof. By means of lemma 2.5 the claim follows along the line of proof for lemma 1.7. □

2.8. Lemma. *Let ν be a volume form on $X(\Gamma)$ and write $\nu(z) = (f(z)+1)\cdot\mu_\Gamma(z)$, then we have*

$$0 \leq c_{\mu_\Gamma,\nu} \leq \frac{2}{\min(\lambda_1, 1/4)}||f||^2_{L^2}, \tag{2.8.1}$$

here λ_1 is the first non zero eigenvalue of Δ_Γ and $||f||^2_{L^2} = \int_X f(z)\overline{f(z)}\mu_\Gamma(z)$.

Proof. We first observe $f(z)$ is a smooth, square integrable function that is orthogonal to the constants with respect to μ. Now with the same arguments as in Lemma 1.8, see also [AU], p.66, we derive the the upper bound

$$\begin{aligned} c_{\mu_\Gamma,\nu} &= \int_{X(\Gamma)\times X(\Gamma)} g_\Gamma(z,w)f(z)\overline{f(w)}\mu_\Gamma(w)\mu_\Gamma(z) \\ &= 2\left(\sum_{m>0}\frac{|a_m|^2}{\lambda_m} + 2\pi\sum_\kappa\int_0^\infty \frac{|h_\kappa(t)|^2}{\frac{1}{4}+t^2}\,\mathrm{d}\,t\right) \\ &\leq 2\left(\frac{1}{\lambda_1}\sum_{m>0}|a_m|^2 + \frac{2\pi}{\frac{1}{4}}\sum_\kappa\int_0^\infty |h_\kappa(t)|^2\,\mathrm{d}\,t\right) \leq \frac{2}{\min(\lambda_1,\frac{1}{4})}||f||^2_{L^2} \end{aligned}$$

and again the lower bound $0 \leq c_{\mu_\Gamma,\nu}$ is obvious. □

3. Intersection numbers of hermitian line bundles

In this section we give an overview of Arakelov theory for arithmetic surfaces, see e.g. [Ga], [Kü], [So].

3.1. Notation. Let K be a number field, $\mathcal{O}_K$ its ring of integers and Σ the set of complex embeddings of K in $\mathbb{C}$. Let X_K be a smooth, projective curve of genus $g \geq 2$ defined over K. An arithmetic surface $\mathcal{X}$ is a regular scheme of dimension 2 together with a projective flat morphism $f : \mathcal{X} \longrightarrow \operatorname{Spec}\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Moreover we assume that the generic fiber X_K of f is geometrically irreducible, i.e., $\mathcal{X}$ is a regular model for X_K over $\operatorname{Spec}\mathcal{O}_K$. We let $\mathcal{X}_\infty$ be the set of complex points of $\mathcal{X}$ of the scheme defined over Spec $\mathbb{Z}$ defined by $\mathcal{X}$, i.e. $\mathcal{X}_\infty = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C})$. Note the complex conjugation F_∞ acts on $\mathcal{X}_\infty$. Finally, by abuse of notation we set

$$\int_{\mathcal{X}_\infty} := \sum_{\sigma\in\Sigma}\int_{X_\sigma(\mathbb{C})}$$

3.2. Arithmetic Picard group. We call a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a hermitian metric h a *hermitian line bundle* and denote it by $\overline{\mathcal{L}} = (\mathcal{L}, h)$. As usual $\mathrm{c}_1(\overline{\mathcal{L}})$ is its first Chern form. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian line bundles on $\mathcal{X}$ and l, m (resp.) be non-trivial, global sections, whose induced divisors on $\mathcal{X}_\infty$ have no points in common. Then, the

arithmetic intersection number $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + (l.m)_{\infty}; \tag{3.2.1}$$

here the contribution at the finite primes $(l.m)_{\mathrm{fin}}$ is defined by Serre's Tor-formula and the contribution at the infinite primes is given by a certain star product $(l.m)_{\infty}$ (cf. the special cases (3.3.1), (3.3.3) and (3.3.2) below). Two hermitian line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if $\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_{\mathcal{X}}, |\cdot|)$. The *arithmetic Picard group*, denoted by $\widehat{\mathrm{Pic}}(\mathcal{X})$, is the group of isomorphy classes of hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ the group structure being given by the tensor product. The principal theorem in Arakelov theory is that the formula (3.2.1) induces a bilinear, symmetric pairing

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \quad \longrightarrow \quad \mathbb{R}.$$

Finally, $\widehat{\mathrm{Pic}}^{0}(\mathcal{X})_{\mathbb{Q}} \subset \widehat{\mathrm{Pic}}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ denotes the subgroup generated by those hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ satisfying $\deg(\mathcal{L}|_{\mathcal{C}_l^{(\mathfrak{p})}}) = 0$ for all irreducible components $\mathcal{C}_l^{(\mathfrak{p})}$ of the fiber $f^{-1}(\mathfrak{p})$ above $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$, and $\mathrm{c}_1(\overline{\mathcal{L}}) = 0$.

3.3. Arithmetic Chow groups. Instead of the arithmetic Picard groups $\widehat{\mathrm{Pic}}(\mathcal{X})$ we could also consider the arithmetic Chow groups $\widehat{\mathrm{CH}}(\mathcal{X})$. The objects of these arithmetic Chow groups are equivalence classes of arithmetic cycles represented by pairs (D, g_D), where D is a divisor on $\mathcal{X}$ and g_D is a Green function for $D_\infty = \coprod_{\sigma \in \Sigma} D_\sigma(\mathbb{C})$. If $\overline{\mathcal{L}}$ is a hermitian line bundle, then the first arithmetic Chern class $\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{CH}}^{1}(\mathcal{X})$ can be represented by any of the pairs $(\mathrm{div}(s), -\log\|s\|^2)$ where s is a section for $\mathcal{L}$. The assignment $\overline{\mathcal{L}} \mapsto \widehat{\mathrm{c}}_1(\overline{\mathcal{L}})$ induces a morphism $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{CH}}^{1}(\mathcal{X})$ that is compatible with the arithmetic intersection pairings in both groups. In particular we have the formulas

$$(\mathrm{div}(s), -\log\|s\|^2).(0, g) = \frac{1}{2}\int_{\mathcal{X}_\infty} g\, \mathrm{c}_1(\overline{\mathcal{L}}), \tag{3.3.1}$$

$$(0, g_1).(0, g_2) = \frac{1}{2}\int_{\mathcal{X}_\infty} g_1 \;\mathrm{dd}^c\, g_2 = \frac{1}{2}\int_{\mathcal{X}_\infty} g_2 \;\mathrm{dd}^c\, g_1, \tag{3.3.2}$$

$$(\mathrm{div}(s), -\log\|s\|^2).(D, 0) = (\mathrm{div}(s), D)_{\mathrm{fin}}, \tag{3.3.3}$$

which are just reformulations of particular cases in (3.2.1).

3.4. Admissible metrics. Let $\mathcal{X}$ be an arithmetic surface. Then, $\mathcal{X}_\infty = \coprod_{\sigma \in \Sigma} X_\sigma(\mathbb{C})$ is a finite union of compact Riemann surfaces. By abuse of notation we call a $(1,1)$-form ν on $\mathcal{X}_\infty$ such that $\nu = \coprod_{\sigma \in \Sigma} \nu_\sigma$, where each ν_σ is a volume form on $X_\sigma(\mathbb{C})$, also a volume form on $\mathcal{X}_\infty$. A hermitian line bundle $\overline{\mathcal{L}}$ is called ν*-admissible* if $\mathrm{c}_1(\overline{\mathcal{L}}) = \deg(\mathcal{L})\nu$.

For a divisor $D = \sum n_P P$ on $\mathcal{X}$ we define $\overline{\mathcal{O}}(D)_\nu = (\mathcal{O}(D), \|\cdot\|_\nu)$, where the metric on the line bundle $\mathcal{O}(D_\sigma)$ on $X_\sigma(\mathbb{C})$ is such that $-\log\|1_{D_\sigma}\|^2_\nu = \sum n_p g_{\nu_\sigma}(z, P)$, where $g_{\nu_\sigma}(z, P)$

is the unique *normalized Green function* for $P_\sigma(\mathbb{C})$ associated with ν_σ (see definition 1.2). Observe $\overline{\mathcal{O}}(D)_\nu$ is a ν-admissible line bundle.

3.5. Logarithmically singular, hermitian line bundles. Let $\mathcal{X}$ be an arithmetic surface such that $\mathcal{X}_\infty \cong \prod_{\sigma\in\Sigma} X(\Gamma_\sigma)$ with fuchsian groups Γ_σ. By abuse of notation we write $\mu_\Gamma = \prod_{\sigma\in\Sigma}\mu_{\Gamma_\sigma}$. It is natural to consider μ_Γ-admissible line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$, i.e., those metrized line bundles equipped with a hermitian metric with $\mathrm{c}_1(\overline{\mathcal{L}}) = \deg(\mathcal{L})\mu_\Gamma$. Such line bundles are examples for logarithmically singular, hermitian line bundles in the sense of [Kü]. It is shown in [Kü] how to modify the arithmetic intersection product in order to obtain an arithmetic intersection product for such singular line bundles. Here we only need that for the particular cases discussed in 3.3 the same formulae (3.3.1), (3.3.2) and (3.3.3) hold

3.6. Theorem. *Let ν be a volume form on $\mathcal{X}_\infty$ and let μ be another volume form or $\mu = \mu_\Gamma$. Then for any μ-admissible hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{O}(D), \|\cdot\|)$ of degree d we have the equality*

$$\overline{\mathcal{O}}(D)_\nu^2 = \overline{\mathcal{L}}^2 + d\int_{\mathcal{X}_\infty} \log\|l\|^2\nu + \frac{d^2}{2}c_{\mu,\nu},$$

where l is a section of $\mathcal{O}(D)$ with divisor D.

Proof. Assume first that μ is a volume form. In the language of arithmetic Chow rings, which we will use for a moment, we write

$$\begin{aligned}\widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(D)_\nu) &= (\mathrm{div}(l), g_\nu(D,z))\\ &= (\mathrm{div}(l), -\log\|l\|^2) + (0, g_\nu(D,z) + \log\|l\|^2))\\ &= \widehat{\mathrm{c}}_1(\overline{\mathcal{L}}) + (0, g_\nu(D,z) - g_\mu(D,z)) + (0, g_\mu(D,z) + \log\|l\|^2);\end{aligned}$$

observe the last term is in fact a vector of constants. Then using formulae (3.3.1), (3.3.2) and (3.3.3), we obtain with the help of lemma 1.7 the equalities

$$\begin{aligned}\overline{\mathcal{O}}(D)_\nu^2 &= \overline{\mathcal{L}}^2 + \frac{d}{2}\int_{\mathcal{X}_\infty}\big(g_\nu(D,z) - g_\mu(D,z)\big)\wedge(\mu(z)+\nu(z)) + d\,\big(g_\mu(D,z) + \log\|l\|^2\big)\\ &= \overline{\mathcal{L}}^2 + d\,a_{\mu,\nu}(D) + \frac{d^2}{2}c_{\mu,\nu} + d(g_\mu(D,z) + \log\|l\|^2)\\ &= \overline{\mathcal{L}}^2 + d\int_{\mathcal{X}_\infty}\log\|l\|^2\nu + \frac{d^2}{2}c_{\mu,\nu}.\end{aligned}$$

Now for μ_Γ we observe first that the same argument of proof works. If D has no support in the cusps we just replace $g_\mu(D,z)$ by $g_{\mu_\Gamma}(D,z)$, else, if the support of D contains a cusp, we have to use $\tilde{g}_{\mu_\Gamma}(D,z;w_0)$ and Lemma 2.7 instead. □

3.7. Canonical metric. If the genus of X is greater than one, then for each σ we have on $X_\sigma(\mathbb{C})$ the *canonical volume form*

$$\nu^\sigma_{\mathrm{can}}(z) = \frac{i}{2g}\sum_j |f^\sigma_j|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $f^\sigma_1(z)\mathrm{d}z$, ... $f^\sigma_g(z)\mathrm{d}z$ is an orthonormal basis of $H^0(X_\sigma(\mathbb{C}),\Omega^1)$ equipped with the natural scalar product. We write ν_{can} for the induced volume form on $\mathcal{X}_\infty$ and for ease of notation we set

$$\overline{\mathcal{O}}(D) = \overline{\mathcal{O}}(D)_{\nu_{\mathrm{can}}}.$$

3.8. Adjunction formula. Due to Arakelov is the observation that there is a unique metric $\|\cdot\|_{\mathrm{Ar}}$ on $\omega_{\mathcal{X}}$ such that for all sections P of $\mathcal{X}$ it holds the adjunction formula

$$\overline{\omega}_{\mathrm{Ar}}.\overline{\mathcal{O}}(P) + \overline{\mathcal{O}}(P)^2 = \log|\Delta_{K|\mathbb{Q}}|. \tag{3.8.1}$$

Moreover $\overline{\omega}_{\mathrm{Ar}} = (\omega_{\mathcal{X}}, \|\cdot\|_{\mathrm{Ar}})$ is a ν_{can}-admissible line bundle. Because of different conventions we recall that

$$\overline{\omega}_{\mathrm{Ar}} = \overline{\omega}_{\mathcal{X}/\mathcal{O}_K,\mathrm{Ar}} \otimes f^*\overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}, \tag{3.8.2}$$

where $\omega_{\mathcal{O}_K/\mathbb{Z}} = \partial^{-1}_{K|\mathbb{Q}}$ is equipped with the natural metric and $\overline{\omega}_{\mathcal{X}/\mathcal{O}_K}$ with its residual metric (see e.g. [MB2], eq. (1.2.1)). Observe the adjunction formula holds for arithmetic surfaces in the sense of 3.1 (see e.g. [La], p.101).

3.9. Lemma. *Let P be a section of $\mathcal{X} \to \operatorname{Spec}\mathcal{O}_K$ and $\mathcal{F}_P$ a fibral divisor with the property*

$$\overline{\omega}_{Ar} \otimes \overline{\mathcal{O}}(P+\mathcal{F}_P)^{2-2g} \in \widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}, \tag{3.9.1}$$

then

$$\overline{\omega}^2_{Ar} = -\,\mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(P)^{2-2g}\right) + (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\,\overline{\mathcal{O}}(P)^2 + (2g-2)^2\mathcal{O}(\mathcal{F}_P)^2,$$

where ht_{NT} *denotes the Neron-Tate height on the Picard group* $\mathrm{Pic}^0(X)$ *of* X*. In particular*

$$\overline{\omega}^2_{Ar} \le (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\,\overline{\mathcal{O}}(P)^2.$$

Proof. These formulae are well-known for semi-stable arithmetic surfaces (see e.g. [Fa], p.410), however the essential ingredients of its proof namely the Faltings Hriljac theorem (see [Hr] theorem 3.1 and [Hr] proposition 3.3) and the adjunction formula (3.8.1) hold also for arithmetic surfaces in the sense of 3.1.

For convenience we recall the proof: There exists a fibral divisor $\mathcal{F}_p$ such that

$$\overline{\omega}_{\mathrm{Ar}} \otimes \overline{\mathcal{O}}(P+\mathcal{F}_P)^{2-2g} \in \widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}, \tag{3.9.2}$$

Then because of the Faltings Hriljac theorem (see [Hr] theorem 3.1 and [Hr] proposition 3.3), the definition of $\widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}$ and the adjunction formula (3.8.1) we have

$$\begin{aligned}
&-\mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(P)^{2-2g}\right)\\
&\qquad=\left(\overline{\omega}_{\mathrm{Ar}}\otimes\overline{\mathcal{O}}(P+\mathcal{F}_P)^{2-2g}\right).\left(\overline{\omega}_{\mathrm{Ar}}\otimes\overline{\mathcal{O}}(P+\mathcal{F}_P)^{2-2g}\right)\\
&\qquad=\overline{\omega}^2_{\mathrm{Ar}}+(4-4g)\overline{\omega}_{\mathrm{Ar}}.\overline{\mathcal{O}}(P)+(2g-2)^2\overline{\mathcal{O}}(P)^2-(2-2g)^2\mathcal{O}(\mathcal{F}_P)^2\\
&\qquad=\overline{\omega}^2_{\mathrm{Ar}}+\left((4g-4)+(2g-2)^2\right)\overline{\mathcal{O}}(P)^2+(4-4g)\log|\Delta_{K|\mathbb{Q}}|-(2-2g)^2\mathcal{O}(\mathcal{F}_P)^2.
\end{aligned}$$

Hence

$$\overline{\omega}^2_{\mathrm{Ar}}=-\mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(P)^{2-2g}\right)+(4g-4)\log|\Delta_{K|\mathbb{Q}}|-4g(g-1)\,\overline{\mathcal{O}}(P)^2+(2g-2)^2\mathcal{O}(\mathcal{F}_P)^2,$$

and with the facts $\mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(P)^{2-2g}\right)\geq 0$ and $\mathcal{O}(\mathcal{F}_P)^2\leq 0$ we derive the desired inequality. □

Note $\mathcal{F}_P$ as above has support in the fibers of bad reduction of $\mathcal{X}$.

4. A new formula for $\overline{\omega}^2_{\mathrm{Ar}}$

Here we consider morphisms of arithmetic surfaces $\boldsymbol{\beta}:\mathcal{X}\to\mathcal{Y}$ and relate $\overline{\omega}^2_{\mathrm{Ar}}$ with arithmetic intersection numbers on $\mathcal{Y}$. Clearly such a formula must contain terms coming from the difference of the metrics on the line bundles in question as well as some contributions coming from the primes of bad reduction of $\mathcal{X}$.

4.1. Proposition. *Let $\boldsymbol{\beta}:\mathcal{X}\to\mathcal{Y}$ be a morphism of arithmetic surfaces. Write d for the degree of the induced morphism $\boldsymbol{\beta}:\mathcal{X}_K\to\mathcal{Y}_K$. Let P be K-rational point on $\mathcal{Y}_K$ and write $\boldsymbol{\beta}^*P=\sum_j b_jS_j$. We assume that all S_j are K-rational points. We denote also by P, resp. S_j, the horizontal divisor induced by the divisor P, resp. S_j. Let ν be a volume form on $\mathcal{X}_\infty$ and let μ be another volume form or $\mu=\mu_\Gamma$. Finally, for $\overline{\mathcal{L}}=(\mathcal{O}(P),\|\cdot\|)\in\widehat{\mathrm{Pic}}(\mathcal{Y})$, we assume that $\boldsymbol{\beta}^*\overline{\mathcal{L}}$ is μ-admissible hermitian line bundle. Under the above assumptions we have*

$$\begin{aligned}
\sum_j b_j\,\overline{\mathcal{O}}(S_j)^2_\nu=&\overline{\mathcal{L}}^2+\int_{\mathcal{X}_\infty}\log\|l\|^2\nu+\frac{d}{2}c_{\mu,\nu}+\sum_j b_j\,\mathcal{O}(\mathcal{G}_j)^2\\
&-\sum_j b_j\,\mathrm{ht}_{NT}\left(\mathcal{O}(S_j)\otimes\boldsymbol{\beta}^*\mathcal{L}^{\otimes-1/d}\right)
\end{aligned}\tag{4.1.1}$$

where l is the unique section of $\boldsymbol{\beta}^*\mathcal{O}(P)$ whose divisor equals the divisor $\boldsymbol{\beta}^*P$ and for each cusp S_j the vertical divisor $\mathcal{G}_j$ is defined in (4.1.2) *below.*

Proof. For each S_j we consider a μ-admissible, hermitian line bundle $\mathcal{L}_j=(\mathcal{O}(S_j),\|\cdot\|)$ together with a fibral divisor $\mathcal{G}_j$ such that

$$\left(\overline{\mathcal{L}}_j\otimes\mathcal{O}(\mathcal{G}_j)\right)\otimes\left(\boldsymbol{\beta}^*\overline{\mathcal{L}}\right)^{\otimes-1/d}\in\widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}.\tag{4.1.2}$$

Let l_j be a section of $\mathcal{L}_j$ with divisor S_j. Then, because of theorem 3.6 we have the equality

$$\overline{\mathcal{O}}(S_j)^2_\nu = \overline{\mathcal{L}}_j^2 + \int_{\mathcal{X}_\infty} \log \|l_j\|^2 \nu + \frac{1}{2} c_{\mu,\nu}. \tag{4.1.3}$$

We further have

$$\begin{aligned}
-\sum_j & b_j \,\mathrm{ht}_{NT}\left(\mathcal{O}(S_j) \otimes \boldsymbol{\beta}^* \mathcal{L}^{\otimes -1/d}\right) \\
&= \sum_j b_j \left(\left(\overline{\mathcal{L}}_j \otimes \mathcal{O}(\mathcal{G}_j)\right) \otimes \left(\boldsymbol{\beta}^* \overline{\mathcal{L}}\right)^{\otimes -1/d} \right)^2 \\
&= \sum_j b_j \left(\overline{\mathcal{L}}_j^2 + 2\mathcal{L}_j.\mathcal{O}(\mathcal{G}_j) + \mathcal{O}(\mathcal{G}_j)^2 - \frac{2}{d} \left(\overline{\mathcal{L}}_j \otimes \mathcal{O}(\mathcal{G}_j)\right).\boldsymbol{\beta}^*\overline{\mathcal{L}} + \frac{1}{d}\overline{\mathcal{L}}^2 \right) \\
&= \sum_j b_j \left(\overline{\mathcal{L}}_j^2 + 2\mathcal{L}_j.\mathcal{O}(\mathcal{G}_j) + \mathcal{O}(\mathcal{G}_j)^2 - \frac{2}{d} \mathcal{O}(\mathcal{G}_j).\boldsymbol{\beta}^*\overline{\mathcal{L}} \right) - \overline{\mathcal{L}}^2 + \sum_j b_j \log \|l_j\|^2 - \log \|l\|^2 \\
&= \sum_j b_j \,\overline{\mathcal{L}}_j^2 + \sum_j b_j \log \|l_j\|^2 - \overline{\mathcal{L}}^2 - \log \|l\|^2 - \sum_j b_j \,\mathcal{O}(\mathcal{G}_j)^2;
\end{aligned}$$

observe in the last equalities we used the relations

$$\begin{aligned}
\sum_j b_j \,\overline{\mathcal{L}}_j.\boldsymbol{\beta}^*\overline{\mathcal{L}} &= \sum_j b_j (S_j, -\log \|l_j\|^2).\boldsymbol{\beta}^*(P, -\log \|l\|^2) \\
&= \left((\sum_j b_j \, S_j, -\log \|l\|^2) + (0, -\sum_j b_j \,\log \|l_j\|^2 + \log \|l\|^2) \right).\boldsymbol{\beta}^*(P, -\log \|l\|^2) \\
&= \boldsymbol{\beta}_*(\sum_j b_j \, S_j, -\log \|l\|^2).(P, -\log \|l\|^2) - d \left(\sum_j b_j \log \|l_j\| - \log \|l\| \right) \\
&= d\,\overline{\mathcal{L}}^2 - d \left(\sum_j b_j \log \|l_j\| - \log \|l\| \right)
\end{aligned}$$

and

$$\mathcal{O}(\mathcal{G}_j).\frac{\boldsymbol{\beta}^*\mathcal{L}}{d} = \mathcal{L}_j.\mathcal{O}(\mathcal{G}_j) + \mathcal{O}(\mathcal{G}_j)^2.$$

Now taking the weighted sum of (4.1.3) the claim follows easily by recollecting the terms in the above equality for the Neron-Tate height in question. □

4.2. Theorem. *Let $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ be a morphism of arithmetic surfaces. Write d for the degree of the induced morphism $\boldsymbol{\beta} : \mathcal{X}_K \to \mathcal{Y}_K$. Let P be K-rational point on $\mathcal{Y}_K$ and write $\boldsymbol{\beta}^* P = \sum_j b_j S_j$. We assume that all S_j are K-rational points. We denote also by P, resp. S_j, the horizontal divisor induced by the divisor P, resp. S_j. Let μ be a volume form or*

$\mu = \mu_\Gamma$. *Then for any* $\overline{\mathcal{L}} = (\mathcal{O}(P), \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{Y})$ *such that* $\boldsymbol{\beta}^*\overline{\mathcal{L}}$ *is* μ*-admissible hermitian line bundle, it holds the equality*

$$\begin{aligned}\overline{\omega}^2_{Ar} =&(4g-4)\log|\Delta_{K|\mathbb{Q}}| - \frac{4g(g-1)}{d}\Big(\overline{\mathcal{L}}^2 + \int_{\mathcal{X}_\infty}\log\|l\|^2\nu_{can} + \frac{d}{2}c_{\mu,\nu_{can}})\Big)\\ &+\frac{1}{d}\sum_j b_j\Big(4g(g-1)\,\mathrm{ht}_{NT}\left(\mathcal{O}(S_j)\otimes\boldsymbol{\beta}^*\mathcal{L}^{\otimes -1/d}\right) - \mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(S_j)^{2-2g}\right)\\ &-4g(g-1)\,\mathcal{O}(\mathcal{G}_j)^2 + (2g-2)^2\,\mathcal{O}(\mathcal{F}_j)^2\Big).\end{aligned} \tag{4.2.1}$$

Here l *is the unique section of* $\boldsymbol{\beta}^*\mathcal{L}$ *whose divisor equals the divisor induced by the divisor* $\boldsymbol{\beta}^*P$ *and* $\mathcal{F}_j$*, resp.* $\mathcal{G}_j$*, are as defined in* (3.9.1)*, resp.* (4.1.2)*.*

Proof. We write $\mathcal{F}_j$ for the vertical divisor determined by S_j by means of formula (3.9.1). Again we write S_j also for the section induced by S_j. Then for each S_j we have because of lemma 3.9

$$\begin{aligned}&\overline{\omega}^2_{\mathrm{Ar}} - (4g-4)\log|\Delta_{K|\mathbb{Q}}|\\ &\quad = -\,\mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(S_j)^{2-2g}\right) + (2g-2)^2\,\mathcal{O}(\mathcal{F}_j)^2 - (4g(g-1))\overline{\mathcal{O}}(S_j)^2.\end{aligned}$$

We now add these equalities for all these K-rational points S_j weighted with the factor b_j/d, which is determined by $\boldsymbol{\beta}^*P = \sum b_jS_j$ and obtain by means of (4.1.1) the equality.

$$\begin{aligned}\overline{\omega}^2_{\mathrm{Ar}} =&(4g-4)\log|\Delta_{K|\mathbb{Q}}| - \frac{4g(g-1)}{d}\Big(\overline{\mathcal{L}}^2 + \int_{\mathcal{X}_\infty}\log\|l\|^2\nu_{\mathrm{can}} + \frac{d}{2}c_{\mu,\nu_{\mathrm{can}}})\Big)\\ &+\frac{1}{d}\sum_j b_j\Big(4g(g-1)\,\mathrm{ht}_{NT}\left(\mathcal{O}(S_j)\otimes\boldsymbol{\beta}^*\mathcal{L}^{\otimes -1/d}\right) - \mathrm{ht}_{NT}\left(\omega\otimes\mathcal{O}(S_j)^{2-2g}\right)\\ &-4g(g-1)\,\mathcal{O}(\mathcal{G}_j)^2 + (2g-2)^2\,\mathcal{O}(\mathcal{F}_j)^2\Big).\end{aligned}$$

□

4.3. Remarks. (i) If in theorem 4.2 the arithmetic surface is semi-stable, then the condition that all S_i are K-rational points can be skipped. Indeed, if L is a finite Galois extension of K so that all S_i are defined over L, then the theorem holds for a minimal desingularisation $\pi : \mathcal{X}' \to \mathcal{X} \times \operatorname{Spec}\mathcal{O}_L$ of the arithmetic surface $f : \mathcal{X}' \to \operatorname{Spec}\mathcal{O}_L$ obtained from $\mathcal{X}$ via base change to L. But then due the facts, that the canonical sheaf $\omega_{\mathcal{X}'} = \pi^*\omega_{\mathcal{X}} \otimes f^*\omega_{\mathcal{O}_L/\mathcal{O}_K}$ (see e.g. [La], p.127) and that the right hand side in formula (4.2.1) is invariant under the Galois group $\mathrm{Gal}(L|K)$, the claim finally follows by dividing the resulting formula by $[L : K]$.

(ii) In many cases the arithmetic surface $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ can be chosen to be the minimal regular model, but we have to stress the fact that in general the arithmetic surface $\mathcal{X}$ is not the minimal regular model. If $\mathcal{X}$ is not the minimal model $\mathcal{X}_{\min}$ of X_K, then there exists a morphism $\pi : \mathcal{X} \to \mathcal{X}_{\min}$ and a vertical divisor $\mathcal{W}$ on $\mathcal{X}$, whose support is in those

fibers which contain a (-1)-curve, such that $\pi^*\omega_{\mathcal{X}_{\min}} = \omega_{\mathcal{X}} \otimes \mathcal{O}(\mathcal{W})$. By means of theorem 4.2 we then have

$$\overline{\omega}^2_{\mathcal{X}_{\min},\mathrm{Ar}} = \pi^*\overline{\omega}^2_{\mathcal{X}_{\min},\mathrm{Ar}} = \overline{\omega}^2_{\mathcal{X},\mathrm{Ar}} + 2\omega_{\mathcal{X}}.\mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W})^2 = \overline{\omega}^2_{\mathcal{X},\mathrm{Ar}} + \sum_{\mathfrak{p}\,\mathrm{bad}} b_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}),$$

with some uniquely determined coefficients $b_{\mathfrak{p}} \in \mathbb{Q}$.

(iii) Recall a prime $\mathfrak{p}$ is said to be bad if the fiber of $\mathcal{X}$ above $\mathfrak{p}$ is reducible. Then obviously the contributions of the fibral intersections to (4.2.1) have only support at the bad primes.

4.4. Definition. Keep the notations as in Theorem 4.2

(i) We call

$$\sum_{\mathfrak{p}\ \mathrm{bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}) = -\frac{2g}{d}\sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 + \frac{2g-2}{d}\sum_j b_j\, \mathcal{O}(\mathcal{F}_j)^2 \tag{4.4.1}$$

the geometric contributions.

(ii) We call

$$a_{\boldsymbol{\beta}} = -\frac{2g}{d}\left(\overline{\mathcal{L}}^2 + \int_{\mathcal{X}_\infty} \log \|l\|^2 \nu_{\mathrm{can}} + \frac{d}{2} c_{\mu,\nu_{\mathrm{can}}}\right)\tag{4.4.2}$$

the analytic contributions.

Observe because of lemma 1.8, resp. lemma 2.8 we have $a_{\boldsymbol{\beta}} \leq b_{\boldsymbol{\beta}}$ where

$$b_{\boldsymbol{\beta}} = -\frac{2g}{d}\left(\overline{\mathcal{L}}^2 + \int_{\mathcal{X}_\infty} \log \|l\|^2 \nu_{\mathrm{can}}\right)\tag{4.4.3}$$

4.5. Corollary. *If in addition to the assumption of theorem 4.2 all the divisors $dS_j - \boldsymbol{\beta}^* P$ are torsion divisors, i.e., if $\mathrm{ht}_{NT}\left(\mathcal{O}(S_j) \otimes \boldsymbol{\beta}^*\mathcal{L}^{\otimes -1/d}\right) = 0$, then*

$$\overline{\omega}^2_{Ar} \leq (2g-2)\left(\log |\Delta_{K|\mathbb{Q}}|^2 + b_{\boldsymbol{\beta}} + \sum_{\mathfrak{p}\ \mathrm{bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p})\right).\tag{4.5.1}$$

Proof. This upper bound for $\overline{\omega}^2_{\mathcal{X},\mathrm{Ar}}$ holds because the Neron-Tate height defines a positive definite quadratic form on $\mathrm{Pic}^0(X)_{\mathbb{Q}}$. □

5. The analytic contributions

The analytical contributions (4.4.3) in theorem 4.2 can be bounded whenever X varies in the discrete family of morphisms to a fixed base curve Y unramified over a fixed set of points $\infty, P_1, ..., P_r$. The condition on $Y \setminus \{\infty, P_1, ..., P_r\}$ being hyperbolic implies that for each embedding $\sigma : K \to \mathbb{C}$ the Riemann surface $Y_\sigma(\mathbb{C})$ is associated to a fuchsian group

$\Gamma_\circ^\sigma$ of the first kind. Of course $\infty, P_1, ..., P_r$ are considered to correspond to cusps or elliptic points.

5.1. Sup norm bounds. Let $\Gamma \subseteq \Gamma_\circ$ be a subgroup of finite index $d = [\Gamma_\circ : \Gamma]$. It is shown by Jorgenson and Kramer in [JK2], theorem 3.1, that there exists a constant $\kappa_\circ$ only depending on $X(\Gamma_\circ)$ such that everyw here on $X(\Gamma)$, whenever its genus satisfies $g \geq 2$ of course, it holds the estimate

$$\nu_{\mathrm{can}} \leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\frac{d}{g}\mu_\Gamma, \tag{5.1.1}$$

where $\operatorname{vol}(X(\Gamma_\circ))$ equals the volume $X(\Gamma_\circ)$ with respect to the hyperbolic measure (2.2.1).

5.2. Theorem. *Let $\Gamma \subseteq \Gamma_\circ$ be a subgroup of finite index and let $f : X(\Gamma) \to X(\Gamma_\circ)$ be the natural morphism of degree $d = [\Gamma : \Gamma_\circ]$. Let L be a line bundle on $X(\Gamma_\circ)$ equipped with either a smooth hermitian or a $\mu_{\Gamma_\circ}$-admissible metric. Let l be a section of L whose divisor* $\operatorname{div}(l)$ *is effective. Then there exists a constant $\kappa \in \mathbb{R}$ independent of Γ, which depends only on* $\log\|l\|$ *as a function on $X(\Gamma_\circ)$, such that*

$$\int_{X(\Gamma)} -\log\|l\|\,\nu_{can} \leq 4\pi\kappa_\circ \sum_{S_j \text{ cusp}} \operatorname{ord}_{f(S_j)}(l)\frac{b_j\log(b_j)}{g} + \frac{d}{g}\kappa, \tag{5.2.1}$$

here ν_{can} is the canonical volume form on $X(\Gamma)$, b_j denotes the ramification index at the cusp S_j and g is the genus of $X(\Gamma)$.

Proof. We first assume that the divisor of l is disjoint to the cusps. Then $\|l\|$ is either a smooth function or a logarithmically-singular function. Thus we may apply inequality (5.1.1) to the integral under consideration and we easily derive the upper bound.

However, if l vanishes at a cusp of $X(\Gamma)$, then the logarithmic singularities of $\log\|l\|^2$ together with the singularities of μ_Γ near the cusps prevent us to apply inequality (5.1.1) directly. The singularties at the elliptic points do not matter.

Without loss of generality we may assume that $X(\Gamma_\circ)$ has only one cusp, which we denote by ∞; the general case of several cusps will not alter the argument of proof.

For a small ε we set

$$X(\Gamma)_\varepsilon = X(\Gamma) \setminus f^*B_\varepsilon(\infty),$$

where $B_\varepsilon(\infty)$ is a small open ball of hyperbolic volume ε around the cusp ∞. Since $0 \leq \nu_{\mathrm{can}} \leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\dfrac{d}{g}\,\mu_\Gamma$ we have

$$\begin{aligned}
\int_{X(\Gamma)_\varepsilon} -\log \|l\|^2\, \nu_{\mathrm{can}} &\leq \int_{X(\Gamma)_\varepsilon} \left[-\log \|l\|^2\right]^+ \nu_{\mathrm{can}} \\
&\leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\frac{d}{g} \int_{X(\Gamma)_\varepsilon} \left[-\log \|l\|^2\right]^+ \mu_\Gamma \\
&\leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\frac{d}{g} \int_{X(\Gamma_\circ)\setminus B_\varepsilon(\infty)} \left[-\log \|l\|^2\right]^+ \mu_{\Gamma_\circ} \\
&\leq \frac{d}{g}\kappa_1(\varepsilon),
\end{aligned}$$

here $[-\log \|l\|^2]^+ = (|-\log \|l\|^2| - \log \|l\|^2)/2$ is the positive part of $-\log \|l\|^2$ and $\kappa_1(\varepsilon)$ is a constant independent of Γ.

It remains to bound the integral over the closure of $f^* B_\varepsilon(\infty)$. If b denotes the width of the cusp ∞, then we have

$$B_\varepsilon(\infty) = \{z = x + iy \in \mathbb{H} \,|\, 0 < x \leq b,\, y > b/\varepsilon\}$$

and for a cusp S_j ramified of order b_j above ∞

$$B_{b_j\varepsilon}(S_j) = \{z = x + iy \in \mathbb{H} \,|\, 0 < x \leq bb_j,\, y > b/\varepsilon\},$$

therefore

$$f^* B_\varepsilon(\infty) = \bigcup_{S_j} B_{b_j\varepsilon}(S_j).$$

We may assume that l vanishes of order 1 at ∞.

For a local coordinate t for the cusp ∞ we shall write $-\log \|l\|^2(t) = -\log |t|^2 + h(t)$, where $|h(t)|$ has at most log-log growth. As before we find a constant $\kappa_2(\varepsilon)$, which is independent of Γ, such that

$$\begin{aligned}
\int_{\overline{B_{b_j\varepsilon}(S_j)}} h(t)\nu_{\mathrm{can}} &\leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\frac{d}{g} \int_{\overline{B_{b_j\varepsilon}(S_j)}} |h(t)|\mu_\Gamma \\
&\leq \kappa_\circ \operatorname{vol}(X(\Gamma_\circ))\frac{d}{g}\frac{b_j}{d} \int_{\overline{B_\varepsilon(\infty)}} |h(t)|\mu_{\Gamma_\circ} \\
&\leq \frac{b_j}{g}\kappa_2(\varepsilon),
\end{aligned}$$

where b_j denotes the ramification index at S_j.

For the remaining term we recall that the local coordinate at the cusp ∞ on the modular curve $X(\Gamma_\circ)$ is given by $\exp(2\pi i z/b)$ with $z = x + iy \in \mathbb{H}$ and $b \in \mathbb{R}$. Therefore $-\log |t|^2 =$

$4\pi y/b$ and

$$\begin{aligned}\int\limits_{\overline{B_{b_j\varepsilon}(S_j)}} -\log|t|^2\,\nu_{\mathrm{can}} &= \int\limits_{\overline{B_{b_j\varepsilon}(S_j)\setminus B_\varepsilon(S_j)}} -\log|t|^2\,\nu_{\mathrm{can}} + \int\limits_{\overline{B_\varepsilon(S_j)}} -\log|t|^2\,\nu_{\mathrm{can}}\\ &\leq \kappa_\circ\frac{1}{g}\iint\limits_{\substack{0<x<b_jb\\ \frac{b}{\varepsilon}<y<\frac{bb_j}{\varepsilon}}}\frac{4\pi}{b}y\,\frac{dxdy}{y^2} + \int\limits_{\overline{B_\varepsilon(S_j)}} -\log|t|^2\,\nu_{\mathrm{can}}\\ &= 4\pi\kappa_\circ\frac{b_j}{g}\log(b_j) + \int\limits_{\overline{B_\varepsilon(S_j)}} -\log|t|^2\,\nu_{\mathrm{can}}\end{aligned}$$

But now with the same considerations as in [JK2], p.1274, we get on

$$\overline{B_\varepsilon(S_j)} = \{z = x+iy\in\mathbb{H}\,|\,0<x\leq bb_j,\, y>bb_j/\varepsilon\}$$

the bound

$$\begin{aligned} g\,\nu_{\mathrm{can}} &= y^2\sum_{r=1}^{g}|f_r(z)|^2\,\frac{dxdy}{y^2}\\ &\leq y^2 e^{-\frac{4\pi y}{bb_j}}\max_{y=\frac{bb_j}{\varepsilon}}\sum_{r=1}^{g}|f_r(z)/e^{-\frac{4\pi y}{bb_j}}|^2\,\frac{dxdy}{y^2}\\ &\leq y^2 e^{-\frac{4\pi y}{bb_j}}e^{\frac{4\pi}{\varepsilon}}\left(\frac{\varepsilon}{bb_j}\right)^2\kappa_\circ\,\frac{dxdy}{y^2}. \end{aligned} \tag{5.2.2}$$

Then (5.2.2) yields

$$\begin{aligned}\int_{\overline{B_\varepsilon(S_j)}} -\log|t|^2\nu_{\mathrm{can}} &\leq \frac{4\pi}{b}\frac{\kappa_\circ}{g}\iint\limits_{\substack{0<x\leq bb_j\\ \frac{b_j}{\varepsilon}<y}} ye^{-\frac{4\pi y}{bb_j}}e^{\frac{4\pi}{\varepsilon}}\left(\frac{\varepsilon}{bb_j}\right)^2 dxdy\\ &= \frac{4\pi}{b}\frac{\kappa_\circ}{g}\frac{b_jb\varepsilon(\varepsilon+4\pi)}{16\pi^2}\\ &\leq \frac{b_j}{g}\kappa_3(\varepsilon)\end{aligned}$$

again with $\kappa_3(\varepsilon)$ a constant independent of Γ.

Now for the general case we take into account that the sum $\sum_{S_j} b_j$ of the ramification indices above the cusps equals d and thus we derive

$$\int_{\overline{f^*B_\varepsilon(\infty)}} -\log\|l\|^2\nu_{\mathrm{can}} \leq 4\pi\kappa_\circ\sum_{S_j}\mathrm{ord}_{f(S_j)}(l)\frac{b_j\log(b_j)}{g} + \frac{d}{g}(\kappa_2(\varepsilon)+\deg(L)\kappa_3(\varepsilon)).$$

Setting $\kappa = \kappa_1(\varepsilon) + \kappa_2(\varepsilon) + \deg(L)\kappa_3(\varepsilon)$ we yield the claimed upper bound in (5.2.1). $\square$

5.3. Corollary. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of arithmetic surfaces as in 0.3. Let $b_{\boldsymbol{\beta}}$ be as in* (4.4.3)*, then there exist constants $\kappa_\circ, \kappa$ independent of $\mathcal{X}$ such that*

$$b_{\boldsymbol{\beta}} \leq -\frac{2g}{d}\overline{\mathcal{L}}^2 + [K:\mathbb{Q}]\left(8\pi\kappa_\circ \sum \frac{b_j \log b_j}{d} + 2\kappa\right). \tag{5.3.1}$$

In particular there exist constants κ_1, κ_2 independent of $\mathcal{X}$ such that

$$b_{\boldsymbol{\beta}} \leq [K:\mathbb{Q}]\left(\kappa_1 \log(\max_j\{b_j\}) + \kappa_2\right).$$

Proof. The inequality (5.3.1) is a direct consequence of Theorem 5.2. The Hurwitz formula implies that g/d is bounded thus the second claim is established. $\square$

6. An upper bounds for the geometric contribution

Here we give general bounds for the quantity $\sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p})$ defined by (4.4.1), i.e., the geometric contributions in the formula of theorem 4.2.

For each bad prime $\mathfrak{p}$ we let

$$\mathcal{X} \times \overline{\mathbb{F}}_{\mathfrak{p}} = \sum_{j=1}^{r_{\mathfrak{p}}} m_j C_j^{(\mathfrak{p})} \tag{6.0.2}$$

be the decomposition in irreducible components and set

$$u_{\mathfrak{p}} = \max_{i,j} |C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}|, \qquad l_{\mathfrak{p}} = \min_{C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})} \neq 0} |C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}|. \tag{6.0.3}$$

We further denote by $c_{\mathfrak{p}}$ the connectivity of the dual graph of $\mathcal{X}(\overline{\mathbb{F}}_{\mathfrak{p}})$, i.e. minimal number of intersection points needed to connect any two irreducible components. With these informations we define

$$b_{\mathfrak{p}} = \left(\sum_{k=1}^{c_{\mathfrak{p}}}\left(\sum_{l=1}^{k}\left(\frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}}\right)^{l-1}\right)^2 + (r_{\mathfrak{p}} - c_{\mathfrak{p}} - 1)\left(\sum_{l=1}^{c_{\mathfrak{p}}}\left(\frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}}\right)^{l-1}\right)^2\right)\frac{u_{\mathfrak{p}}}{l_{\mathfrak{p}}^2}. \tag{6.0.4}$$

6.1. Proposition. *Let $\mathcal{G}_j$ be as in* (4.1.2)*, then its component $\mathcal{G}_j^{(\mathfrak{p})}$ in the fibre above a bad prime $\mathfrak{p}$ satisfies*

$$-\left(\mathcal{G}_j^{(\mathfrak{p})}\right)^2 \leq b_{\mathfrak{p}}$$

Proof. After possibly renumbering the irreducible components and adding rational multiples of full fibers, we may assume $0 \neq \mathcal{G}_j^{(\mathfrak{p})} = \sum_{k=2}^{r_\mathfrak{p}} n_k C_k^{(\mathfrak{p})}$ with all $n_k \geq 0$ and $n_1 = 0$.

Let $W = \{ C_j^{(\mathfrak{p})} \}$ be the set of all irreducible components of the fibre above $\mathfrak{p}$ and set

$$U_0 = \{C_j^{(\mathfrak{p})} \in W \mid n_j = 0\}$$
$$V_0 = W \setminus U_0.$$

Then we define recursively

$$U_{k+1} = \{C_j^{(\mathfrak{p})} \in V_k \mid \exists\, C_i^{(\mathfrak{p})} \in U_k \,\text{with}\, C_j^{(\mathfrak{p})}.C_i^{(\mathfrak{p})} > 0\}$$
$$V_{k+1} = V_k \setminus U_{k+1}.$$

Since the fibre above $\mathfrak{p}$ is connected, the subsets $U_k \subset W$ determine a disjoint decomposition of W. In fact this decomposition has at most $c_\mathfrak{p} + 1$ disjoint sets.

For all $C_j^{(\mathfrak{p})} \in U_1$ we have

$$n_j \leq \frac{1}{l_\mathfrak{p}}.$$

Indeed there exists a $C_l^{(\mathfrak{p})} \in U_0$ with $C_j^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} > 0$ and with this component we obtain the upper bound

$$\deg\left(\mathcal{L}|_{C_l^{(\mathfrak{p})}}\right) = d\left(S_j + \mathcal{G}_j^{(\mathfrak{p})}\right).C_l^{(\mathfrak{p})} = d\left(S_j.C_l^{(\mathfrak{p})} + \sum_{k=2}^{r_\mathfrak{p}} n_k C_k^{(\mathfrak{p})}.C_l^{(\mathfrak{p})}\right) \leq d. \tag{6.1.1}$$

Recall that $S_j.C_l^{(\mathfrak{p})} \geq 0$ and that by construction all the summands $n_k C_k^{(\mathfrak{p})}.C_l^{(\mathfrak{p})}$ in (6.1.1) are non negative. Thus we derive the bound for n_j.

For all $C_j^{(\mathfrak{p})} \in U_2$ we have

$$n_j \leq \frac{1}{l_\mathfrak{p}}\left(1 + \frac{u_\mathfrak{p}}{l_\mathfrak{p}}\right).$$

Indeed there exists a $C_l^{(\mathfrak{p})} \in U_1$ with $C_j^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} > 0$ and with this component we obtain the upper bound

$$\begin{aligned} dn_j l_\mathfrak{p} \leq dn_j C_j^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} \leq dS_j.C_l^{(\mathfrak{p})} + d\sum_{\substack{k=2\\k\neq l}}^{r_\mathfrak{p}} n_k C_k^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} &= \deg\left(\mathcal{L}|_{C_l^{(\mathfrak{p})}}\right) - dn_l\, C_l^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} \\ &\leq d\left(1 + \frac{u_\mathfrak{p}}{l_\mathfrak{p}}\right), \end{aligned}$$

since $0 \leq \deg\left(\mathcal{L}|_{C_l^{(\mathfrak{p})}}\right) \leq d$ and $C_l^{(\mathfrak{p})}.C_l^{(\mathfrak{p})} < 0$.

Repeating this procedure we get for the remaining coefficients n_j for $C_j^{(\mathfrak{p})} \in U_k$ the bound

$$n_j \leq \frac{1}{l_\mathfrak{p}} \sum_{l=1}^{k} \left(\frac{u_\mathfrak{p}}{l_\mathfrak{p}}\right)^{l-1}.$$

So we obtain

$$\begin{aligned} -\left(\mathcal{G}_j^{(\mathfrak{p})}\right)^2 &= -\sum_{j,k=2}^{r_\mathfrak{p}} n_j n_k C_j^{(\mathfrak{p})}.C_k^{(\mathfrak{p})} \\ &\leq -\sum_{j=2}^{r_\mathfrak{p}} n_j^2 C_j^{(\mathfrak{p})}.C_j^{(\mathfrak{p})} \\ &\leq \sum_{\substack{U_k \subset W \\ U_k \neq U_0}} \#U_k \cdot \left(\sum_{l=1}^{k} \left(\frac{u_\mathfrak{p}}{l_\mathfrak{p}}\right)^{l-1}\right)^2 \frac{u_\mathfrak{p}}{l_\mathfrak{p}^2} \\ &\leq b_\mathfrak{p}. \qquad (6.1.2) \end{aligned}$$

For the last inequality we used $1 \leq \#U_k, \#U_{c_\mathfrak{p}} \leq r_\mathfrak{p} - c_\mathfrak{p}$ and $\frac{u_\mathfrak{p}}{l_\mathfrak{p}} \geq 1$. □

6.2. Proposition. *Let $\mathcal{G}_j$, resp. $\mathcal{F}_j$, be as in* (4.1.2), *resp.* (3.9.1). *If $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1$ is a Galois cover, then we have*

$$\mathcal{G}_j^2 = \mathcal{F}_j^2.$$

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a Galois cover of arithmetic surfaces, then there exists a line bundle $\mathcal{L}$ on $\mathcal{Y}$ with:

$$\omega_\mathcal{X}^{\deg(f)} \cong f^*\mathcal{L}. \qquad (6.2.1)$$

Indeed, by assumption $\mathcal{Y} = \mathcal{X}/G$ for a finite group G. Let $s \in H^0(\mathcal{X}, \omega_\mathcal{X})$, then also $s^\sigma \in H^0(\mathcal{X}, \omega_\mathcal{X})$ for all $\sigma \in G$ and therefore

$$g = \prod_{\sigma \in G} s^\sigma \in H^0(\mathcal{X}, \omega_\mathcal{X}^{\otimes |G|}).$$

However, since $g^\sigma = g$ for all $\sigma \in G$, we have $\mathcal{O}(\mathrm{div}(g)) = f^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on $\mathcal{Y}$. Hence

$$\omega_\mathcal{X}^{\otimes |G|} \cong \mathcal{O}(\mathrm{div}(g)) \cong f^*\mathcal{L},$$

from which we deduce the claim (6.2.1) by using the fact $|G| = \deg(f)$.

Since any line bundle on $\mathbb{P}^1$ is of the form $\mathcal{O}(1)^{\otimes e} \otimes \mathcal{O}(\mathcal{F})$, where the fibral divisor $\mathcal{F}$ is a finite sum of complete fibers, we have $\mathcal{F}_j = \mathcal{G}_j$. □

6.3. Theorem. *Let $a_{\mathfrak{p}}$ be as in* (4.4.1) *and $b_{\mathfrak{p}}$ as in* (6.0.4)*, then for all primes of bad reduction we have*

$$a_{\mathfrak{p}} \leq 2g\, b_{\mathfrak{p}}.$$

If in addition $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ is a Galois cover and if $\mathcal{Y} = \mathbb{P}^1$, then it holds the stronger inequality

$$a_{\mathfrak{p}} \leq 2\, b_{\mathfrak{p}}.$$

Proof. In order to verify the first claim it suffices to bound the terms involving $\mathcal{O}(\mathcal{G}_j)^2$, since $\mathcal{O}(\mathcal{F}_j)^2 \leq 0$ for all j. Upper bounds have been given in proposition 6.1. Now summing up over all j yield the first claim.

In order to verify the second claim we first observe that by proposition 6.2 the quantity $a_{\mathfrak{p}}$ only involves $\mathcal{G}_j$ and then we use again proposition 6.1. □

7. Explicit calculations for the modular curves $X_0(N)$

We first aim to give explicit formulas for the coefficients a_p in (4.4.1) for the modular curves $X_0(N)$, this in turn proves Theorem II from the introduction.

7.1. Preliminaries. The curves $X_0(N)$ have a model over $\mathbb{Q}$ and its complex valued points correspond to the compact Riemann surfaces $\Gamma_0(N) \setminus (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$, where $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid c \equiv 0 \mod N \}$. The index in $\Gamma(1)$ is given by the formula

$$d = [\Gamma(1) : \Gamma_0(N)] = \prod_{p|N} (p+1).$$

From now on N will be a square free integer with $(6, N) = 1$ having at least two different prime factors, because then the cusps are also rational points [Og]. The Manin-Drinfeld theorem assures that any divisor of degree zero with support in the cusps is a torsion divisor.

Recall that the genus of $X_0(N)$ in our cases is given by

$$g = 1 + \frac{d}{12} - \frac{1}{4} \prod_{p|N} \left(1 + \left(\frac{-1}{p} \right) \right) - \frac{1}{3} \prod_{p|N} \left(1 + \left(\frac{-3}{p} \right) \right) - \frac{1}{2} \sigma_0(N),$$

where $(\frac{\cdot}{p})$ are the Legendre symbols and $\sigma_0(N)$ denotes the number of divisors of N. Therefore for a small $\varepsilon > 0$ we have since N is square free

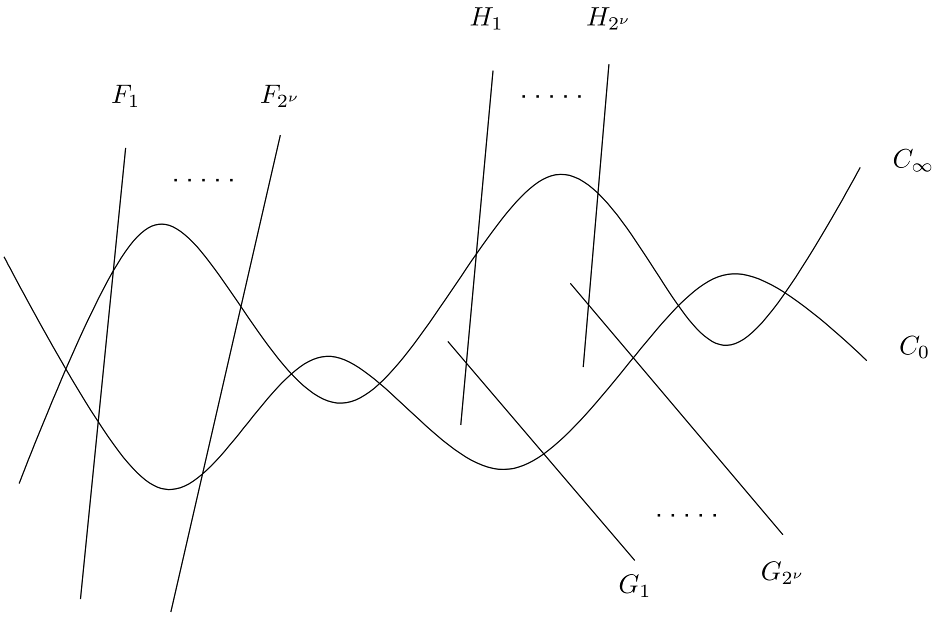
$$\frac{12(g-1)}{d} = 1 + O(N^{-\varepsilon}). \tag{7.1.1}$$

7.2. Minimal model. The minimal model $\mathcal{X}_0(N)$ over $\operatorname{Spec}\mathbb{Z}$ has been determined by Deligne and Rapoport to be as follows: The curve $X_0(N)$ is smooth over $\mathbb{Z}[1/N]$. If $p|N$, then the scheme $\mathcal{X}_0(N) \otimes \mathbb{Z}/p\mathbb{Z}$ is reduced and singular over $\mathbb{Z}/p\mathbb{Z}$. We write $N = pM = pq_1 \cdot \ldots \cdot q_\nu$ and set $Q = \prod_{i=1}^{\nu}(q_i + 1)$ and define

$$u = \begin{cases} 1 & \text{if } p \equiv 7 \text{ or } 11 \mod 12 \text{ and all } q_i \equiv 1 \mod 4, \quad i = 1 \ldots \nu \\ 0 & \text{else} \end{cases}$$

$$v = \begin{cases} 1 & \text{if } p \equiv 5 \text{ or } 11 \mod 12 \text{ and all } q_i \equiv 1 \mod 3, \quad i = 1 \ldots \nu \\ 0 & \text{else} \end{cases}$$

The fiber $\mathcal{X}_0(N) \otimes \mathbb{Z}/p\mathbb{Z}$ is the union of two copies C_0, C_∞ of $\mathcal{X}_0(M) \otimes \mathbb{Z}/p\mathbb{Z}$ crossing transversely in certain supersingular points and some chains of projective lines connecting the remaining supersingular points with those of the other copy. More precisely, if $u = 1$ then there are 2^ν projective lines F_i, if $v = 1$ then there are 2^ν pairs G_i, H_i of projective lines and if $u = v = 0$ there are no such projective lines. All theses projective lines have self-intersection number -2. The intersection behavior between these irreducible components is given by the figure below.



In this figure all intersections are transversal, i.e., all intersection multiplicities are equal to 1. Finally we mention the formula

$$C_0.C_\infty = d\frac{p-1}{12(p+1)} - 2^\nu\left(\frac{u}{2} + \frac{v}{3}\right).$$

The natural morphism $\boldsymbol{\beta}: X_0(N) \to X(1)$ extends to a morphism of $\boldsymbol{\beta}: \mathcal{X}_0(N) \to \mathcal{X}(1)$. Its degree equals d and the local degrees at the bad fibres are $d/(p+1)$, $pd/(p+1)$ or zero at the irreducible components C_∞, C_0 or any other.

There are two cusps 0 and ∞ and their associated horizontal divisors are disjoint.

7.3. Lemma. *With the notation as in* (4.1.2) *we have on* $\mathcal{X}_0(N)$ *the formula*

$$\sum_j b_j \mathcal{O}(\mathcal{G}_j)^2 = -12 \sum_{p|N} \frac{p}{p^2-1} \log p,$$

Proof. Recall that the global sections line bundle $\mathcal{L} = \boldsymbol{\beta}^* \mathcal{O}(1)$ are given by modular forms of weight 12. Because of the Manin-Drinfeld theorem there exists for any cusp S_j a integral modular form g_j of some weight $12e$ which vanish on $X_0(N)$ only at S_j.

We can consider the above equality prime by prime. We put

$$\mathcal{D}_0 = C_0 + \frac{1}{2}\sum_\nu F_\nu + \frac{2}{3}\sum_\mu G_\mu + \frac{1}{3}\sum_\rho H_\rho;$$

where if some of the components F_ν, G_μ, H_ρ do not exist, then these are set to be the zero divisors.

For a cusp S_j we have

$$\operatorname{div}(g_j^{1/e})\big|_{\mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z}} \equiv d\,(S_j + \alpha_j \mathcal{D}_0) \mod \mathcal{X}_0(N)\otimes \mathbb{Z}/p\mathbb{Z},$$

with

$$\alpha_j = \frac{12}{d(p-1)}\left(1-(p+1)\,S_j.C_\infty\right).$$

Indeed the relations $\deg\operatorname{div}(g_j^{1/e})\big|_{F_\nu} = \deg\operatorname{div}(g_j^{1/e})\big|_{G_\mu} = \deg\operatorname{div}(g_j^{1/e})\big|_{H_\rho} = 0$ are easily checked and also the remaining relation

$$\frac{d}{p+1} = \deg\operatorname{div}(g_j^{1/e})\big|_{C_\infty} = d\,\left(S_j + \alpha_j\mathcal{D}_0\right).C_\infty = d\,\left(S_j.C_\infty + \alpha_j d\frac{p-1}{12(p+1)}\right);$$

holds. Set $\mathcal{D}_\infty = \mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z} - \mathcal{D}_0 = C_\infty + \frac{1}{2}\sum_\nu F_\nu + \frac{1}{3}\sum_\mu G_\mu + \frac{2}{3}\sum_\rho H_\rho$. From $\mathcal{D}_0.(\mathcal{D}_0+\mathcal{D}_\infty) = 0$, we deduce

$$\mathcal{D}_0^2 = -\mathcal{D}_0.\mathcal{D}_\infty = -\mathcal{D}_0.C_\infty = -d\frac{p-1}{12(p+1)}.$$

Then as $\mathcal{G}_j^{(p)} = \alpha_j\,\mathcal{D}_0$ and as $S_j.C_\infty \in \{0,1\}$, the last equality in turn implies

$$(\mathcal{G}_j^{(p)})^2 = \alpha_j^2\,\mathcal{D}_0^2 = -\frac{12}{d}\left(\frac{1}{p^2-1} + S_j.C_\infty\right).$$

Now summing up over all cusps S_j yields the identity

$$\begin{aligned}\sum_{S_j} b_j \mathcal{O}(\mathcal{G}_j^{(p)})^2 &= -\frac{12}{d}\sum_{S_j} b_j\left(\frac{1}{p^2-1}+S_j.C_\infty\right)\log(p)\\ &= -\frac{12}{d}\left(\frac{d}{p^2-1}+\frac{d}{p+1}\right)\log(p)\\ &= -12\left(\frac{p}{p^2-1}\right)\log p.\end{aligned}$$

Finally summing up over all the primes of bad reduction leads to the claim. □

7.4. Lemma. *With the notation as in* (3.9.1) *we have on* $\mathcal{X}_0(N)$ *the formulae*

$$\sum_{p|N}\sum_{S_j} b_j\mathcal{O}(\mathcal{F}_j^{(p)})^2 = -3\sum_{p|N}\frac{p+1}{p-1}\log p = -3\log N - \sum_{p|N}\frac{6}{p-1}\log p,$$

Proof. The proof of this formula is similar to those of lemma 7.3, we refer to [AU], Proposition 4.2.1, p.63. □

7.5. Proposition. *With the notation as in* (4.4.1) *we have on* $\mathcal{X}_0(N)$ *the estimate*

$$\sum_{p \text{ bad}} a_p \log p = -\frac{1}{2}\log N + O(N^{-\varepsilon}\log(N))$$

Proof. By means of the lemmata 7.3 and 7.4 we have the equality

$$\begin{aligned}\sum_{p\text{ bad}} a_p\log p &= -\frac{2g}{d}\sum_j b_j\mathcal{O}(\mathcal{G}_j)^2 + \frac{(2g-2)}{d}\sum_{S_j} b_j\mathcal{O}(\mathcal{F}_j)^2\\ &= \frac{2}{d}\left(\sum_{p|N}\left(12g\frac{p}{p^2-1}-(g-1)\frac{6}{p-1}\right)\log p - 3(g-1)\log N\right)\\ &= -\frac{1}{2}\log N + O(N^{-\varepsilon}\log(N)).\end{aligned}$$

□

7.6. Theorem. *Let N be a square free integer having at least two different prime factors and $(N,6)=1$. Let $\mathcal{X}_0(N)$ be the minimal regular model of the modular curve $X_0(N)$ and g_N its genus. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \leq (16\pi\kappa_\circ - 1)g_N\log(N) + O(g_N) \tag{7.6.1}$$

where $\kappa_\circ \in \mathbb{R}$ is an absolute constant independent of N.

Proof. To ease notation we write g for g_N. The minimal model $\boldsymbol{\beta} : \mathcal{X}_0(N) \to \mathcal{X}(1)$ together with its cusps S_j satisfies the assumption theorem 4.2. We take P to be the cusp $\infty \in X(1)(\mathbb{Q})$. We may assume that $\overline{\mathcal{L}}$ is μ_Γ admissible.

We present now for the positive terms on the right hand side bounds or estimates.

Since $K = \mathbb{Q}$ the relative discriminant equals 1, thus first term vanishes.

By the genus formula (7.1.1) for $X_0(N)$ we deduce for the second term

$$-\frac{4g(g-1)}{d}\overline{\mathcal{L}}^2 = O(g).$$

For the integral we use Theorem 5.2 and the fact that the cusp 0 has the maximal possible width N, i.e. $b_{\max} = N$,

$$\begin{aligned}\frac{4g(g-1)}{d}\int_{X(\Gamma_0(N))} - \log \|l\|\, \nu_{\mathrm{can}} &\leq \frac{4g(g-1)}{d}\left(4\pi\kappa_\circ \sum_{S_j} \frac{b_j \log(b_j)}{g} + \frac{d}{g}\kappa\right)\\ &\leq (g-1)16\pi\kappa_\circ \log(N) + O(g)\end{aligned}$$

We can neglect the contribution with the quantity $c_{\mu_\Gamma,\nu_{\mathrm{can}}}$, as by Lemma 2.8 this is a positive real number.

There will be no positive contribution of the terms made with Neron-Tate heights. This is due to the theorem of Manin-Drinfeld and the positivity of the Neron-Tate height.

The remaining contributions we have calculated in Proposition 7.5

$$(2g-2)\sum_{p \text{ bad}} a_p \log p = -g\log N + O(g).$$

□

Since $\mathcal{X}_0(N)$ is a semi-stable model of $X_0(N)$ the arithmetic self-intersection number of its dualizing sheaf is positive, this implies immediately the following results.

7.7. Corollary. *Keep the previous notation. Then*

$$\sum_j \frac{b_j}{d}\,\mathrm{ht}_{NT}\left(\omega \otimes \mathcal{O}(S_j)^{2-2g}\right) = O(g_N \log(N)).$$

and

$$c_{\mu_\gamma,\nu_{can}} = O(\log(N)/g_N),$$

□

8. Bounds for Fermat curves F_p

We now consider the Fermat curves $F_n : x^n + y^n = z^n$. The morphism $\boldsymbol{\beta} : F_N \to \mathbb{P}^1$ given by $(x : y : z) \mapsto (x^n : z^n)$ determines a Galois covering with group $(\mathbb{Z}/n\mathbb{Z})^2$. Since $\boldsymbol{\beta}$ has the three ramification points $0, 1, \infty$, it is a Belyi morphism. The ramification indices are all equal to n. In contrast to the previous examples, we proceed with the identification $\mathbb{P}^1 \setminus \{0, 1, \infty\} = \Gamma(2) \setminus \mathbb{H}$. The principal congruence subgroup $\Gamma(2)$ is a free group on two generators A, B. Let

$$\Gamma_n = \left\{ \gamma = A^{e_1} B^{f_1} \cdots A^{e_r} B^{f_r} \in \Gamma(2) \;\middle|\; \sum e_i = \sum f_j = n \right\},$$

then $\Gamma(2)/\Gamma_n \cong (\mathbb{Z}/n\mathbb{Z})^2$. Therefore $F_n = X(\Gamma_n)$ and $\boldsymbol{\beta}$ is induced by the natural morphism $\Gamma_n \setminus \mathbb{H} \to \Gamma(2) \setminus \mathbb{H}$. The cusps are defined over $\mathbb{Q}(\zeta_n)$ and the group of cuspidal divisors on F_n with respect to the uniformization given by Γ_n modulo rational equivalence is a torsion subgroup, in fact its structure is determined in [Roh].

Let $\mathcal{X}$ be the desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p + y^p = z^p$ with prime exponent p (see [Mc]). It is smooth outside the primes above p. If $\mathfrak{p}$ be a prime above p, then it is shown in [Mc] that $\mathcal{X}(\overline{\mathbb{F}}_\mathfrak{p})$ has at most

$$r_\mathfrak{p} \leq 4 + p(p-3)/2 \tag{8.0.1}$$

irreducible components $C_l^{(\mathfrak{p})}$, which are all isomorphic to $\mathbb{P}^1$, and

$$u_\mathfrak{p} := \max_{l,m} \left| C_l^{(\mathfrak{p})}.C_m^{(\mathfrak{p})} \right| \leq p \text{ and } l_\mathfrak{p} = \min_{C_l^{(\mathfrak{p})}.C_m^{(\mathfrak{p})} \neq 0} \left| C_l^{(\mathfrak{p})}.C_m^{(\mathfrak{p})} \right| = 1 \tag{8.0.2}$$

Furthermore for the connectivity of the dual graph of $\mathcal{X}$ we have $c_\mathfrak{p} = 3$, thus

$$b_\mathfrak{p} \leq \left(\sum_{k=1}^{3} \left(\sum_{l=1}^{k} p^{l-1} \right)^2 + \frac{p(p-3)}{2} \left(\sum_{l=1}^{3} p^{l-1} \right)^2 \right) p \leq \frac{p^7}{2}$$

8.1. Theorem. *Let $\mathcal{X}$ be the desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p + y^p = z^p$ with prime exponent p (see* [Mc]*). Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \leq (2g_p - 2) \left(\log |\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_p) : \mathbb{Q}](\kappa_1 \log p + \kappa_2) + g_p\, p^7 \log p \right)$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ are absolute constants independent of p.

Proof. Because of the previous discussion, we are allowed to apply corollary 4.5 with $P = \infty \in \mathbb{P}^1$. Then corollary 5.3 and theorem 6.3 finish the proof. $\square$

9. Bounds for modular curves $X(N)$

We now consider the modular curves $X(N)$, where $N = p^k m$ and we assume $p > 3$ is a prime and m is coprime to $6p$. These curves have a model over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of unity and its complex valued points correspond to the compact Riemann surfaces $\Gamma(N) \setminus (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$, where $\Gamma(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \}$. The natural morphism $X(N) \to X(1)$ is a Galois covering with three branch points. The ramification index above the cusp $\infty \in X(1)$ equals N.

Regular models $\mathcal{X}(N)$ over $\operatorname{Spec} \mathbb{Z}[\zeta_N]$ have been obtained by Deligne and Rapoport via normalisation and by Katz and Mazur via moduli interpretation. In the latter case the scheme $\mathcal{X}(N)$ is regular and it is smooth over $\mathbb{Z}[\zeta_N, 1/N]$. If $\mathfrak{p}|p$, then the fiber $\mathcal{X}(N) \otimes \overline{\mathbb{F}}_{\mathfrak{p}}$ is the union of

$$r_{\mathfrak{p}} = p^k + p^{k-1}$$

irreducible components crossing in the

$$s_{\mathfrak{p}} = \frac{p-1}{24} m^2 \phi(m) \prod_{q|m} \left(1 + \frac{1}{q}\right)$$

supersingular points of $\mathcal{X}(N/p^k) \otimes \overline{\mathbb{F}}_{\mathfrak{p}}$ with a maximal multiplicity $m_{\mathfrak{p}} = p^{2k-2}$. From this description we deduce

$$u_{\mathfrak{p}} \leq (r_{\mathfrak{p}} - 1) m_{\mathfrak{p}} s_{\mathfrak{p}}, \quad l_{\mathfrak{p}} \geq s_{\mathfrak{p}}, \quad c_{\mathfrak{p}} = 1$$

and therefore

$$b_{\mathfrak{p}} \leq \frac{(r_{\mathfrak{p}} - 1)^2 m_{\mathfrak{p}}}{s_{\mathfrak{p}}}$$

Finally we recall that the natural morphism $\boldsymbol{\beta} : \mathcal{X}(N) \to \mathcal{X}(1)$ is Galois covering.

9.1. Theorem. *Let $\mathcal{X}(N)$ be a regular model of the modular curve $X(N)$ as above. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \leq (2g_N - 2) \left(\log |\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_N) : \mathbb{Q}](\kappa_1 \log N + \kappa_2) + 2 \sum_{\mathfrak{p} \supset (N)} \frac{(r_{\mathfrak{p}} - 1)^2 m_{\mathfrak{p}}}{s_{\mathfrak{p}}} \log \operatorname{Nm} \mathfrak{p} \right),$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ are absolute constants independent of N.

Proof. Because of the previous discussion, we are allowed to apply corollary 4.5 with $P = \infty \in \mathcal{X}(1)$. Now by means of corollary 5.3 and theorem 6.3 the claim follows $\square$

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